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A Linear Matrix Inequality Approach to Decentralized Control of Distributed Parameter Systems

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Abstract

In this paper preliminary results in the use of linear matrix inequalities for the decentralized control of distributed parameter systems is presented. The class of systems being considered are those that can be expressed as multidimensional systems. It is shown that linear matrix inequalities can be used to provide tractable solutions to this problem; the conditions are in general conservative, but are computationally attractive and lead to controllers which have a decentralized structure.

1 Introduction

Many systems consist of similar units which directly interact with their nearest neighbors. Even when these units have tractable models and interact with their neighbors in a simple and predictable fashion, when viewed as a whole the resulting system often displays rich and complex behavior. There are many examples of such systems, including:

- Automobiles on a freeway; during periods of congestion, drivers are typically concerned with the position and velocity of the vehicle directly in front and directly behind them. Even though a driver's response in these situations may be predictable and easy to model, the overall behavior of the vehicles on the freeway is very complex, and is prone to many types of instabilities . This has led research into automated highway systems in the hope of increasing vehicle throughput and eliminating traffic instabilities [9].
- Formation flight of unmanned aerial vehicles; in these applications, unmanned vehicles are flown in close formation in order to increase the effective aspect ratio of the vehicles and thus reduce drag [12]. The identical vehicles are coupled to their nearest neighbors aerodynamically, and any control system being sought must take this coupling into account to ensure that disturbances are not amplified as they propagate through the system.
- Certain classes of partial differential equations; many PDEs are derived by considering the interaction of an infinite number of infinitesimal elements interacting with their nearest neighbors; examples include the deflection of beams, plates, and membranes, and the temperature distribution of thermally conductive materials.

An important aspect of many of these systems is that sensing and actuation capabilities exist at every unit. In the examples above, this is clearly the case for vehicle platoons and aerial vehicle systems; with the rapid advances in micro electro-mechanical actuators and sensors, one may control the vibrations of plates by instrumenting them with a large number of distributed actuators and sensors as well. If one attempts to control these systems using standard control design techniques, severe limitations will be quickly encountered as most optimal control techniques cannot handle systems of high dimension and with a large number of inputs and outputs. In addition, it is typically not feasible to control these systems with centralized schemes, as these require high levels of connectivity, impose a substantial computational burden, and are typically more sensitive to failures and modeling errors than decentralized schemes.

In order for any optimal control technique to be successful, the structure of the system must be exploited in order to obtain tractable algorithms. As shall be demonstrated, the types of problems discussed above may be cast as multidimensional optimization problems; see [11], [7], [2], and the references therein, for further motivation.

The class of systems described in this paper fall under the class of spatially invariant systems, as described in [1]. Equivalently, the systems considered in this paper can be captured as a linear fractional transformation of a structured, multidimensional operator and a constant matrix. One of the main features of this approach is that casting problems in this form naturally leads to partially decentralized strategies; the amount of decentralization, in turn, can be tuned by the designer, and there thus exists a tradeoff between the amount of decentralization and the achievable performance of the control system. The derived conditions take the form of computationally tractable linear matrix inequalities (LMIs) [3].

The paper is organized as follows. We begin with some mathematical preliminaries in Section 2, followed by an example in Section 3 which motivates the system class

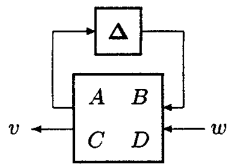

Figure 1: LFT representation

introduced in Section 2. The problem formulation is presented in Section 4, followed by a discussion of how LMIs may be used to provide constructive, sufficient conditions in Section 5. This is followed by a discussion on controller implementation in Section 6. Due to space limitations, the proofs and numerical examples have been omitted; the interested reader may browse through the author's web site for further details and simulations.

2 Preliminaries

We will consider vector valued trajectories which are functions of $L+1$ independent variables, $u = u(t, s_1, \cdots, s_L)$, where t is a non-negative integer, and the s_i are integers. The reason for this asymmetry is that t is used to denote a temporal dimension, while the s_i are used to denote spatial dimensions. This will be made clear later on in the text.

We will use s to denote the L-tuple $(s_1, \cdots, s_L)$. Given u and v, define the following inner product:

$$\langle u, v\rangle := \sum_{t=0}^{\infty} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_L=-\infty}^{\infty} u(t,\mathrm{s})^* v(t,\mathrm{s}) \tag{1}$$

and the corresponding induced norm $\|u\| := \sqrt{\langle u, u\rangle}$.

We are interested in multidimensional systems which can be expressed in the following form:

$$\begin{bmatrix} x^0(t+1, s_1, \cdots, s_L) \\ x^1(t, s_1+1, \cdots, s_L) \\ x^{\bar{1}}(t, s_1-1, \cdots, s_L) \\ \vdots \\ x^{\bar{L}}(t, s_1, \cdots, s_L-1) \end{bmatrix} = \begin{bmatrix} A^{0,0} & A^{0,1} & A^{0,\bar{1}} \ldots & A^{0,\bar{L}} \\ A^{1,0} & A^{1,1} & A^{1,\bar{1}} \ldots & A^{1,\bar{L}} \\ A^{\bar{1},0} & A^{\bar{1},1} & A^{\bar{1},\bar{1}} \ldots & A^{\bar{1},\bar{L}} \\ & & \vdots & \\ A^{\bar{L},0} & A^{\bar{L},1} & A^{\bar{L},\bar{1}} \ldots & A^{\bar{L},\bar{L}} \end{bmatrix} x(t,\mathrm{s})$$

$$+ \begin{bmatrix} B^{0,0} \\ B^{1,0} \\ B^{\bar{1},0} \\ \vdots \\ B^{\bar{L},0} \end{bmatrix} w(t,\mathrm{s});$$

$$v(t,\mathrm{s}) = \begin{bmatrix} C^{0,0} C^{0,1} C^{0,\bar{1}} \ldots C^{0,\bar{L}} \end{bmatrix} x(t,\mathrm{s}) + Dw(t,\mathrm{s}),$$

The above set of equations define an input output map from w to v when the initial state of the system is set to zero: $x(0, \mathrm{s}) \equiv 0$. This may be captured as a linear fractional transformation (LFT) between a structured operator and a constant matrix, as depicted in Figure 1. In particular, define the following shift operators

$$(\boldsymbol{\lambda}_0 u)(t, s_1, \cdots s_L) := u(t-1, s_1, \cdots, s_L); \tag{2}$$
$$(\boldsymbol{\lambda}_i u)(t, s_1, \cdots s_i, \cdots, s_L) := u(t, s_1, \cdots, s_i - 1, \cdots, s_L)$$

and the following structured operator:

$$\boldsymbol{\Delta} := \mathbf{diag}(\boldsymbol{\lambda}_0 I, \boldsymbol{\lambda}_1 I, \boldsymbol{\lambda}_1^{-1} I, \cdots, \boldsymbol{\lambda}_L^{-1} I) \tag{3}$$

partitioned conformably to x. The input-output map from w to v, denoted $\mathbf{G}$, may be written as

$$\mathbf{G} := D + C\boldsymbol{\Delta}(I - A\boldsymbol{\Delta})^{-1}B \tag{4}$$

when the inverse of $(I - A\boldsymbol{\Delta})$ exists; when this inverse exists, the realization is said to be *well posed.*

For example, consider the following equations in one independent variable s_1:

$$x^1(s_1+1) = x^{\bar{1}}(s_1) + w(s_1); x^{\bar{1}}(s_1-1) = x^1(s_1).$$

These equations implicity constrain w to be zero for all s_1; equivalently, the inverse of $\begin{bmatrix} \boldsymbol{\lambda}_1^{-1} & -1 \\ -1 & \boldsymbol{\lambda}_1 \end{bmatrix}$ does not exist. Defining $M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $(\boldsymbol{\Delta}, M)$ is said to be a realization of system $\mathbf{G}$, as per equations (4) and (2).

3 Motivating Example

The class of problems being explored will be motivated through the use of a simple physical example. Consider the temperature distribution of a homogenous, thin plate, $U = U(t, \mathrm{x}, \mathrm{y})$. It is required to regulate the temperature of the plate in the presence of heat disturbance $r = r(t, \mathrm{x}, \mathrm{y})$ using a distributed actuator that can source and sink heat, $u = u(t, \mathrm{x}, \mathrm{y})$. The sensor variables $y = y(t, \mathrm{x}, \mathrm{y})$ are obtained using a distributed sensor which measures the temperature U, subject to additive noise $n = n(t, \mathrm{x}, \mathrm{y})$. These types of problems arise in high quality xerography applications; the quality of a printed page is affected by temperature fluctuations on a charged plate (variable U), which are caused by disturbances created by the material being printed on a previous page (variable r). Feedback is being considered as a means to reduce the fluctuations in U.

Ignoring any actuator and sensor dynamics, and assuming that the plate is large enough to be modeled as infinite, the simplified, normalized partial differential equations which govern the system described above are the following:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial^2 U}{\partial \mathrm{x}^2} + \frac{\partial^2 U}{\partial \mathrm{y}^2} + r + u \\ y &= U + n. \end{aligned} \tag{5}$$

A model of the form (2) may be obtained by spatially and temporally discretizing equations (5), as described below.

If the sensors and actuators are not distributed, but consist of large discrete arrays, there is a natural spatial discretization step Δs. Assuming that the control algorithm will have a digital implementation, there is also an associated sampling time Δt. Using a simple Euler discretization results in the following set of equations:

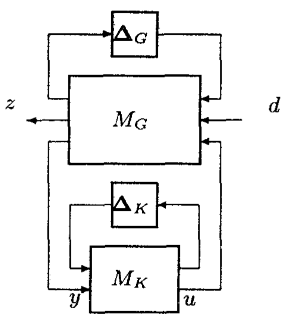

Figure 2: Problem Formulation

$$\frac{1}{\Delta t}(1-\boldsymbol{\lambda}_t)U = \frac{1}{\Delta s^2}\left(\boldsymbol{\lambda}_x^{-1} + \boldsymbol{\lambda}_x + \boldsymbol{\lambda}_y^{-1} + \boldsymbol{\lambda}_y - 4\right)U + r + u$$
$$y = U + n, \tag{6}$$

where the shift operators $\boldsymbol{\lambda}_t, \boldsymbol{\lambda}_x, \boldsymbol{\lambda}_y$ are defined as

$$(\boldsymbol{\lambda}_t U)(t, \mathrm{x}, \mathrm{y}) := U(t - \Delta t, \mathrm{x}, \mathrm{y}) \tag{7}$$
$$(\boldsymbol{\lambda}_x U)(t, \mathrm{x}, \mathrm{y}) := U(t, \mathrm{x} - \Delta \mathrm{s}, \mathrm{y}) \tag{8}$$
$$(\boldsymbol{\lambda}_y U)(t, \mathrm{x}, \mathrm{y}) := U(t, \mathrm{x}, \mathrm{y} - \Delta \mathrm{s}). \tag{9}$$

In order to avoid introducing new independent variables, we have slightly abused notation by taking t, x, and y to be integers and using short form $U(t, \mathrm{x}, \mathrm{y})$ to denote $U(t\Delta t, \mathrm{x}\Delta\mathrm{s}, \mathrm{y}\Delta\mathrm{s})$. Note that the discretization steps in the x and y spatial dimensions could have been chosen differently, if desired. Also note that the heat equation could have been derived from first principles assuming discrete spatial and temporal independent variables, leading to some interesting and non trivial issues associated with the right way to discretize a system (analogous to the many ways of converting a continuous time system to a discrete time system, such as bilinear transformations, sample and hold, Euler discretization, etc.). Equations (6) can be expressed in the form of (2) by defining $w = (r, n, u)$, $v = (U, y)$, and an appropriate x.

4 Problem Formulation

As suggested by the above example, the variables w of a given system may consist of the exogenous disturbances (denoted d) and the control variables (denoted u); thus $w = (d, u)$. Similarly, v consists of the signals which are required to be small (denoted z) and the sensor variables (denoted y); thus $v = (z, y)$. The control design objective is to find a relation between y and u (the controller), such that the resulting closed loop system is stable and the map from d to z is small. In order to make this precise, we need the notions of stability and performance, described below.

Let the system of equations (2) be well posed. We have the following definition of internal stability:

Definition 1 *Let $(\boldsymbol{\Delta}, M)$ be a realization of system* $\mathbf{G}$, *as per equation (2). The realization is said to be* internally stable *if there exists a number α such that for all initial $x(0,\cdot)$ satisfying $\|x(0,\cdot)\| \le 1$, $\|x\| \le \alpha$ when $w \equiv 0$.*

We characterize the performance of a system $\mathbf{G}$ as follows:

Definition 2 *System* $\mathbf{G}$ *is said to be* contractive *if $\|\mathbf{G}w\| \le \|w\|$ for all w; it is said to be* strictly contractive *if $\|\mathbf{G}w\| \le \beta\|w\|$ for some $\beta < 1$.*

We are now in a position to formulate the control design problem. Let $(\boldsymbol{\Delta}_G, M_G)$ be a realization for the given system $\mathbf{G}$, where the inputs are partitioned into $w = (d, u)$ and the outputs into $v = (z, y)$. We are required to find a realization $(\boldsymbol{\Delta}_K, M_K)$ and corresponding system $\mathbf{K}$ such that the closed loop system, as depicted in Figure 2, is internally stable and strictly contractive. We are thus seeking a controller which has the *same structure as the plant* which will internally stabilize the system and which will result in an energy gain from input to output which is less than one. We will later see that the imposed structure on the controller results in a very attractive implementation.

Returning to the plate example, the exogenous disturbances are denoted by $d = (r, n)$, while $z = (U, u)$; we want the temperature U and the control effort u to be small in the presence of heat disturbances r and sensor noise n. M_G and M_K are constant matrices, where M_G is given (the plant) and M_K is to be determined (the controller). $\boldsymbol{\Delta}_G$ and $\boldsymbol{\Delta}_K$ are diagonal operators of the form

$$\boldsymbol{\Delta} = \mathbf{diag}\left(\boldsymbol{\lambda}_t I, \boldsymbol{\lambda}_x I, \boldsymbol{\lambda}_x^{-1} I, \boldsymbol{\lambda}_y I, \boldsymbol{\lambda}_y^{-1} I\right). \tag{10}$$

Several points are worth noting. The first is that in the absence of the operators $\boldsymbol{\lambda}_x, \boldsymbol{\lambda}_x^{-1}, \boldsymbol{\lambda}_y$, and $\boldsymbol{\lambda}_y^{-1}$, the block diagram of Figure 2 captures the standard, one dimensional plant-controller feedback interconnection, where M_G and M_K are the corresponding state space matrix realizations. The second is that in the absence of the operators $\boldsymbol{\lambda}_x^{-1}$ and $\boldsymbol{\lambda}_y^{-1}$, the block diagram captures the feedback interconnection of two multidimensional systems written in Roesser [10] state space form; the operators $\boldsymbol{\lambda}_x^{-1}$ and $\boldsymbol{\lambda}_y^{-1}$ allow one to explicitly capture spatially anti-causal input-output maps without which one could not capture physically meaningful relationships such as $u(t, \mathrm{x}, \mathrm{y}) = \mathrm{y}(\mathrm{t}, \mathrm{x}+1, \mathrm{y}) + \mathrm{y}(\mathrm{t}, \mathrm{x}-1, \mathrm{y})$. Finally, note that $\boldsymbol{\lambda}_t^{-1}$ is not included, since it is reasonable to restrict the plant and controller to be temporally causal.

The example in this section motivates the general definitions established in Section 2. $\boldsymbol{\lambda}_0$ is used to denote $\boldsymbol{\lambda}_t$, and the spatial shift operators are denoted $\boldsymbol{\lambda}_i$ (for the plate example, $L = 2$ and $\boldsymbol{\lambda}_1 := \boldsymbol{\lambda}_x$, $\boldsymbol{\lambda}_2 := \boldsymbol{\lambda}_y$).

5 Synthesizing Controllers

We present below a constructive method for guaranteeing that the closed loop system is both internally stable and strictly contractive. This is achieved as follows; a characterization of internally stable and strictly contractive is obtained for a realization $(\boldsymbol{\Delta}, M)$ which closely parallels the μ framework [8]. An alternate condition is then obtained via bilinear transformations which eliminates the need for working with the inverse operators $\boldsymbol{\lambda}_i^{-1}$, and so that existing synthesis conditions may be readily applied to our problem formulation almost directly. The resulting controller data is then transformed back and a realization for a suitable controller $\mathbf{K}$ obtained.

Given realization $(\mathbf{\Delta}, M)$ (the closed loop system), define the following set of complex matrices $D\mathbf{\Delta}$:

$$D\mathbf{\Delta} := \{\Delta_\mu^D = (\Delta^D, \Delta_F) : |\lambda_0| \le 1, \lambda_i = \lambda_{\bar{i}}^*, \\ |\lambda_i| = 1, \bar{\sigma}(\Delta_F) \le 1\}, \tag{11}$$

where $\Delta^D = (\lambda_0 I, \lambda_1 I, \lambda_{\bar{1}} I, \cdots, \lambda_{\bar{L}} I)$, and Δ_F is a rectangular matrix of dimension which will be clear in a moment. We then have the following result:

Theorem 1 *$(\mathbf{\Delta}, M)$ is internally stable and strictly contractive if and only if*

$$\sup_{\Delta_\mu^D \in D\mathbf{\Delta}} \bar{\sigma}\left((I - M\Delta_\mu^D)^{-1}\right) < \infty. \tag{12}$$

Note that the above condition is very similar to a structured singular value test [8], the main difference being that the λ_i are restricted to be on the unit circle, and that the $\lambda_{\bar{i}}$ are restricted to be the complex conjugate transpose of λ_i. In order to eliminate the $\lambda_{\bar{i}}$ from the above condition, we may perform a bilinear transformation on the λ_i and λ_0 as follows:

$$\lambda_0 =: \frac{s_0 - 1}{s_0 + 1},\ \lambda_i =: \frac{s_i - 1}{s_i + 1},\ \lambda_{\bar{i}} =: \frac{s_i + 1}{s_i - 1}. \tag{13}$$

Define

$$C\mathbf{\Delta} := \{\Delta_\mu^C = (\Delta^C, \Delta_F) : \mathrm{Re}(s_0) \ge 0, \mathrm{Re}(s_i) = 0, \\ \bar{\sigma}(\Delta_F) \le 1\}, \tag{14}$$

where $\Delta^C = \mathbf{diag}(s_0 I, \cdots, s_L I)$. One may readily construct matrix $\hat{M}$ such that $\hat{M}\Delta^C = M\Delta^D$; the details are omitted (see [13], for example). The following corollary follows immediately:

Corollary 1 *$(\mathbf{\Delta}, M)$ is internally stable and strictly contractive if and only if*

$$\sup_{\Delta_\mu^C \in C\mathbf{\Delta}} \bar{\sigma}\left((I - \hat{M}\Delta_\mu^C)^{-1}\right) < \infty. \tag{15}$$

Define the following set of scaling matrices

$$\mathcal{Z} := \{Z = Z^* : Z = \mathbf{diag}(Z_0, Z_1, \cdots, Z_L), Z_0 > 0\}. \tag{16}$$

The corollary can be used to provide the following sufficient condition for internal stability and strict contractiveness:

Theorem 2 *Let* $\hat{M} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$, *and define* $P = \begin{bmatrix} \hat{C}^*\hat{C} & \hat{C}^*\hat{D} \\ \hat{D}^*\hat{C} & \hat{D}^*\hat{D} - I \end{bmatrix}$. *Then $(\mathbf{\Delta}, M)$ is internally stable and strictly contractive if there exists a $Z \in \mathcal{Z}$ such that*

$$P + \begin{bmatrix} \hat{A}^*Z + Z\hat{A} & Z\hat{B} \\ \hat{B}^*Z & 0 \end{bmatrix} < 0 \tag{17}$$

Note that the above is also necessary when $L = 0$, and is in fact the Kalman-Yakubovich-Popov lemma. So far we have only discussed *analysis*; matrix $\hat{M}$ is both a function of M_G (which is given), and M_K (which is what we are searching for). Synthesis conditions may be readily obtained, however, by invoking the LMI $\mathcal{H}_\infty$ solution of [6], which is also based on the KYP lemma. In fact, the synthesis conditions look identical to those in [6] for the continuous time $\mathcal{H}_\infty$ problem, with the exception that the scaling matrices are now $\mathcal{Z}$, and the *coupling conditions* between the two $\mathcal{H}_\infty$ LMIs are only present for the Z_0 scale. Due to space limitations, the details are omitted.

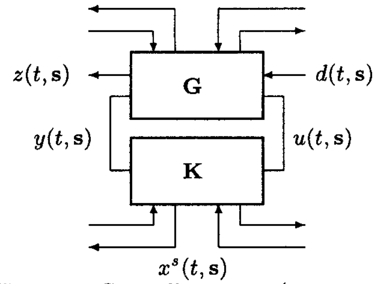

Figure 3: Controller at $\mathbf{s} = (s_1, \cdots, s_L)$

The LMI conditions in [6] do not yield M_K directly, but rather give $\hat{M}_K$. We next discuss how one may obtain M_K from $\hat{M}_K$, and use this matrix to implement the resulting control strategy.

6 Implementation of Control Strategy

Let $\hat{M}_K$ be of the form $\begin{bmatrix} \hat{A}_{tt} & \hat{A}_{ts} & \hat{B}_t \\ \hat{A}_{st} & \hat{A}_{ss} & \hat{B}_s \\ \hat{C}_t & \hat{C}_s & \hat{D} \end{bmatrix}$ consistent with the partition of Δ^C into temporal and spatial components. We provide below an algorithm for constructing M_K which leads to a very desirable controller implementation:

1. Define $\mathcal{X} := \{X = X^* : X = \mathbf{diag}(X_1, \cdots, X_L)\}$, consistent with the spatial components of Δ^C. Solve, if possible, the following LMI:
$$\hat{A}_{ss}^* X + X\hat{A}_{ss} < 0 \tag{18}$$
2. Perform a state transformation on $\hat{M}_K$ so that the elements of the X which solves the above equation are of the form $X_i = \mathbf{diag}(\Sigma_i, -\Sigma_{\bar{i}})$, where Σ and $\Sigma_{\bar{i}}$ are positive definite.
3. Perform the following inverse bilinear transformations: $\lambda_0 := \frac{s_0-1}{s_0+1}$, $\lambda_i := \frac{s_i-1}{s_i+1}$, $\lambda_{\bar{i}} := \frac{s_i+1}{s_i-1}$, where the multiplicity of each λ_i and $\lambda_{\bar{i}}$ is dictated by the size of Σ_i and $\Sigma_{\bar{i}}$. This results in an M_K of the form $\begin{bmatrix} A_{tt} & A_{ts} & B_t \\ A_{st} & A_{ss} & B_s \\ C_t & C_s & D \end{bmatrix}$, where A_{ss} satisfies inequality $A_{ss}^*\Sigma A_{ss} - \Sigma < 0$, and $\Sigma = \mathbf{diag}(\Sigma_1, \Sigma_{\bar{1}}, \cdots, \Sigma_L, \Sigma_{\bar{L}})$ is positive definite.

A realization for a controller which internally stabilizes the given system and results in a strictly contractive map is thus given by $(\mathbf{\Delta}_K, M_K)$, where $\mathbf{\Delta}_K$ is of the form of equation 3, and the multiplicity of each $\boldsymbol{\lambda}_i$ and $\boldsymbol{\lambda}_i^{-1}$ is dictated by the size of the Σ_i and $\Sigma_{\bar{i}}$. The above construction, however, also ensures that

$$(I - \mathbf{\Delta}_s A_{ss})^{-1} = \sum_{k=0}^{\infty} (\mathbf{\Delta}_s A_{ss})^k, \qquad (19)$$

where $\mathbf{\Delta}_s$ is the spatial portion of $\mathbf{\Delta}_K$. Why this is a desirable property is explained below.

The resulting controller may be implemented as in equation (2): at each location s, a state space controller is implemented with state $x^0(t, \mathrm{s})$, sensor input $w = y(t, \mathrm{s})$, and control output $v = u(t, \mathrm{s})$. Furthermore, a controller is connected to its *nearest neighbors* via $x^{\mathrm{s}}(t, \mathrm{s})$ and $\mathbf{\Delta}_s^{-1} x^{\mathrm{s}}(t, \mathrm{s})$, where x^{s} is the spatial portion of x.; in this way, information can be passed anywhere along the spatial dimensions. This is depicted in Figure 3.

One of the potential difficulties with this implementation is that it assumes an instantaneous transfer of information along the spatial variables x^{s}. In particular, we must solve the equation $x^{\mathrm{s}}(t, \mathrm{s}) = \mathbf{\Delta}_s(A_{st} x^t(t, \mathrm{s}) + A_{ss} x^{\mathrm{s}}(t, \mathrm{s}) + B_s y(t, \mathrm{s}))$ at time t, *over all space.* While this may seem difficult, it may be readily achieved by gating the data and clocking it to its nearest neighbors. In particular, for each time t, define

$$(x^{\mathrm{s}}(t, \mathrm{s}))_{k+1} = \mathbf{\Delta}_s(A_{st} x^t(t, \mathrm{s}) + A_{ss}(x^{\mathrm{s}}(t, \mathrm{s}))_k + B_s y(t, \mathrm{s})). \qquad (20)$$

By virtue of equation (19), the above sequence in k converges to the desired $x^{\mathrm{s}}(t, \mathrm{s})$ at each location in space. Thus the controller may be implemented with a two rate scheme; one clock is used to update the state, while another faster clock is used to pass spatial information along the spatial variables.

Note that the level of decentralization can be controlled by lumping variables together in the modeling stage. In particular, consider a system with one spatial dimension x. Let $y(t, \mathrm{x})$ corresponds to the sensor values at spatial location x. One could define $\tilde{y}(t, \tilde{\mathrm{x}}) = (y(t, 3\tilde{\mathrm{x}}-1), y(t, 3\tilde{\mathrm{x}}), y(t, 3\tilde{\mathrm{x}}+1))$, similarly for u, d, and z, and construct a model with variables $\tilde{d}$, $\tilde{z}$, $\tilde{u}$, and $\tilde{y}$. The resulting model can still be written as a multidimensional system, and the previous arguments apply. In the limit, one would obtain a totally centralized scheme. While this approach improves the achievable performance, this is done at the expense of higher connectivity, increased real-time computational burden, and in terms of design, an extremely large optimization problem.

7 Concluding remarks

One of the attractive features of this approach is that the models can be expressed as linear fractional transformations; LFTs have been used extensively in robust control and μ theory [8], and it is likely that many of those results and techniques have applicability to the problems being considered here. Another strength of this approach is that the solutions take the form of LMIs, which have been shown to be a very powerful tool in control theory [3]; it is suspected that the flexibility of LMIs will allow many of the existing results in optimal control to be directly extended to the class of problems being considered in this paper.

The results in this paper are currently being expanded in several directions. One such direction is to allow for spatially *varying* systems [5]; examples include any system with finite boundary conditions. Another direction is to provide direct synthesis conditions in discrete time as opposed to performing bilinear transformations and solving the problems using continuous time techniques, and to provide synthesis techniques where there is a mix of continuos and discrete dynamics [4].

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